(a) Show that $\sqrt{\frac{\mu_0}{\varepsilon_0}}$ has the dimension of resistance

* $F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$ $\quad \frac{df}{dl} = \frac{\mu_0}{4\pi}\frac{2I_1I_2}{r}$

$[\varepsilon_0] = \frac{[q^2]}{[L^2][MLT^{-2}]} = [M^{-1}L^{-3}T^4I^2]$ $\quad \left[\frac{\mu_0}{\varepsilon_0}\right] = \left[\frac{MLT^{-2}I^{-2}}{M^{-1}L^{-3}T^4I^2}\right]$

$[\mu_0] = \frac{[MLT^{-2}][L]}{[L][I^2]} = [MLT^{-2}I^{-2}]$ $\quad = [M^2L^4T^{-6}I^{-4}]$

$R = [ML^2T^{-3}I^{-2}]$

Therefore $\left[\sqrt{\frac{\mu_0}{\varepsilon_0}}\right] = [R]$

*(i) $[R] = \frac{[\varphi]}{[I]} = ML^2T^{-3}I^{-2}$ $\quad [E] = \frac{[\varphi]}{[d]} = MLT^{-3}I^{-1}$

$\frac{[E]}{[H]} = ML^2T^{-3}I^{-2} = [R]$ $\quad \frac{[H]}{[d]} = \frac{[I]}{[d^2]}$

Since $\vec{\nabla}\times\vec{H} = \vec{J}$

Therefore $[H] = IL^{-1}$

(b) The electrostatic pot at a pt. (x,y) is given by $V = 2x+4y$ V. Find the electrostatic en density in J/m^3.

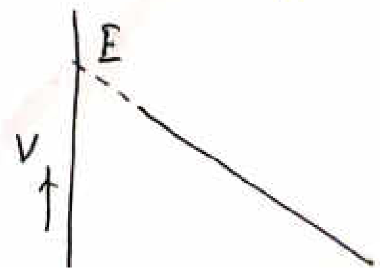
$\vec{E} = -\vec{\nabla}V = -(2\hat{i}+4\hat{j})$

$u = \frac{1}{2}\varepsilon_0\vec{E}\cdot\vec{E} = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\times 8.85\times10^{-12}\times 20 = 8.85\times10^{-12}$ J/m^3

(c) E.M.F. is the open circuited voltage across a battery, then how e.m.f. can be measured in a closed circuit?

$V = E - Ir$

The intercept which the graph makes on the V-axis gives the value of e.m.f.



Scanned by CamScanner

(d) Verify the magnetic vector potential $\vec{A}$ due to uniform magnetic field $\vec{B}$ is given by $\vec{A} = -\frac{1}{2}(\vec{r} \times \vec{B})$.

$$\vec{A} = -\frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= -\frac{1}{2}\left\{\hat{i}(yB_z - zB_y) + \hat{j}(zB_x - xB_z) + \hat{k}(xB_y - yB_x)\right\}$$

now
$$\vec{\nabla}\times\vec{A} = -\frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yB_z - zB_y & zB_x - xB_z & xB_y - yB_x \end{vmatrix}$$

$$= -\frac{1}{2}\left\{\hat{i}(-B_x - B_x) + \hat{j}(-B_y - B_y) + \hat{k}(-B_z - B_z)\right\}$$

$$= \vec{B}.$$

(e) In a certain region the el field in Sph pol Coord is given by $\vec{E} = (a\sin\theta\,\hat{r} + b\cos\theta\,\hat{\theta})$, a, b Const. Prove that the charge density responsible for the field is

$$\rho = \frac{(2a\sin^2\theta + b\cos 2\theta)}{r\sin\theta}$$

$$\rho = \varepsilon_0\vec{\nabla}\cdot\vec{E} = \varepsilon_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, E_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}E_\phi\right]$$

$$\rho = \varepsilon_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, b\cos\theta)\right]$$

$$= \varepsilon_0\left[\frac{a}{r^2}(2r\sin\theta) + \frac{b}{2r\sin\theta}\,2\cos 2\theta\right]$$

$$= \varepsilon_0\left[\frac{2a\sin^2\theta + b\cos 2\theta}{r\sin\theta}\right]$$

(f) The variation of electrostatic pot is given by

$$V(r) = V_0 \frac{e^{-\alpha r}}{r} \quad (V_0 \text{ is Cnst})$$

Obtain corresponding el fld and charge density. When this potential resembles Coulomb's potential?

$$\vec{E} = -\hat{r}\frac{\partial V}{\partial r} = -V_0 \hat{r}\left[-\alpha \frac{e^{-\alpha r}}{r} - \frac{e^{-\alpha r}}{r^2}\right]$$

$$= \hat{r}\frac{V_0}{r} e^{-\alpha r}\left(\alpha + \frac{1}{r}\right)$$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 E\right) = \frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \rho = \frac{\varepsilon}{r^2}\frac{d}{dr}\left(r^2 E\right) = \frac{\varepsilon_0 V_0}{r^2}\frac{d}{dr}\left[r e^{-\alpha r}\left(\alpha + \frac{1}{r}\right)\right]$$

$$= \frac{\varepsilon_0 V_0}{r^2}\left[\alpha e^{-\alpha r} - \alpha^2 r e^{-\alpha r} - \alpha e^{-\alpha r}\right]$$

$$\Rightarrow \rho = -\varepsilon_0 V_0 \frac{\alpha^2}{r} e^{-\alpha r} \quad \text{for } r \neq 0.$$

As $r \to 0$, V approaches that for a point charge located at $r = 0$.

(g) A current distn gives rise to the mag vector pot $\vec{A}(x,y,z) = 2xy^2\hat{i} - x^2y\hat{j} - 3xyz\hat{k}$. Find the corresponding mag fld $\vec{B}$ at $(-1,2,1)$.

$$\vec{B}(x,y,z) = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2 & -x^2y & -3xyz \end{vmatrix} = \hat{i}(-3xz+0) + \hat{j}(0+3yz) + \hat{k}(-2xy - 4xy)$$

$$\Rightarrow \vec{B}(-1,2,1) = 3\hat{i} + 6\hat{j} + 12\hat{k}$$

(h) In a certain region of space el fld is given by $\vec{E} = \hat{j}E_0\cos(\omega t - kx)$. Using diffl form of Faraday's law find the corresponding mag fld $\vec{B}$.

Scanned by CamScanner

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \frac{\partial \vec{B}}{\partial t} = -\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_0 \cos(\omega t - kx) & 0 \end{vmatrix} = -\hat{k} E_0 k \sin(\omega t - kx)$$

Integrating $\vec{B} = \hat{k}\frac{E_0 k}{\omega} \cos(\omega t - kx)$

(i) Show that the ratio of the dim of el field $\vec{E}$ and the mag fld $\vec{H}$ is same as that of resistance.

$$[R] = \frac{[\phi]}{[I]} = ML^2T^{-3}I^{-2} \qquad [E] = \frac{[\phi]}{[d]} = MLT^{-3}I^{-1}$$

Since $\vec{\nabla}\times\vec{H} = \vec{J}$ $\qquad \frac{[H]}{[d]} = \frac{[I]}{[d^2]}$

Therefore $[H] = IL^{-1}$ and $\frac{[E]}{[H]} = ML^2T^{-3}I^{-2} = [R]$.

(j) The el. st. field $\vec{E}$ in $(x-y)$ plane is given by $\vec{E} = 2ax\,\hat{i} + by\,\hat{j}$. What charge density is responsible for the fld

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \text{ or } (2a+b) = \frac{\rho}{\epsilon_0}, \text{ Therefore } \rho = \epsilon_0(2a+b)$$

(k) The el fld in a region of space is given by $\vec{E} = 8x\,\hat{i} - 4y\,\hat{j} - 4z\,\hat{k}$. Find the eq^n of lines of force in the plane $(z=0)$. $\quad E_x = 8x,\ E_y = -4y\ \&\ E_z = -4z$

In the plane $z=0$, $E_z = 0$ $\qquad$ $x-y$ plane

The slope of the lines of force is $\frac{E_y}{E_x} = \frac{dy}{dx} = \frac{-4y}{8x} = -\frac{y}{2x}$

or $\frac{2dy}{y} = -\frac{dx}{x}$

integrating

$\Rightarrow 2\ln y = -\ln x + \ln c$

$\Rightarrow xy^2 = c$

(l) 'Ampere's circuital law is bound to fail for non-steady currents'. — Justify the statement.

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

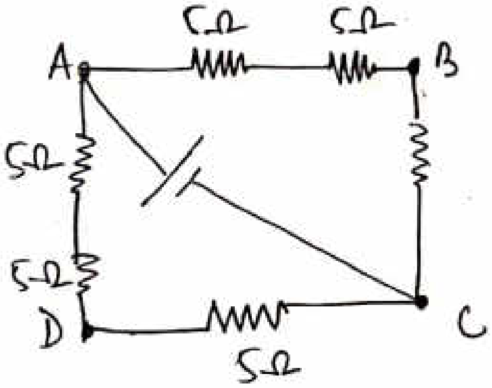
$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 \vec{\nabla} \cdot \vec{J} \text{ or } \vec{\nabla} \cdot \vec{J} = 0 \quad \text{—(1)}$$

From Continuity eq^n $\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$ i.e. $\vec{\nabla} \cdot \vec{J} \neq 0$ —(2)

for charge density varying with time.

Therefore $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ is valid for steady current only.

(m) Find the voltage across AB of the circuit given below:



eqv resistance $R = \frac{15}{2}\,\Omega$

$i = \frac{20}{15} = \frac{4}{3}$ amp.

ABC $\Rightarrow i_1 = \frac{i}{2} = \frac{2}{3}$ amp.

$V_{AB} = \frac{2}{3} \times 10 = 6.67$ ~~ab~~

(n) The el fld in a certain region is given by $\vec{E} = 5r^3\hat{r}$. Prove that charge contained within a sph. surface of radius a centered at the origin is $20\pi\epsilon_0 a^5$.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \Rightarrow \frac{1}{r^2}\frac{d}{dr}(r^2 E_r) = \frac{\rho}{\epsilon_0}$$

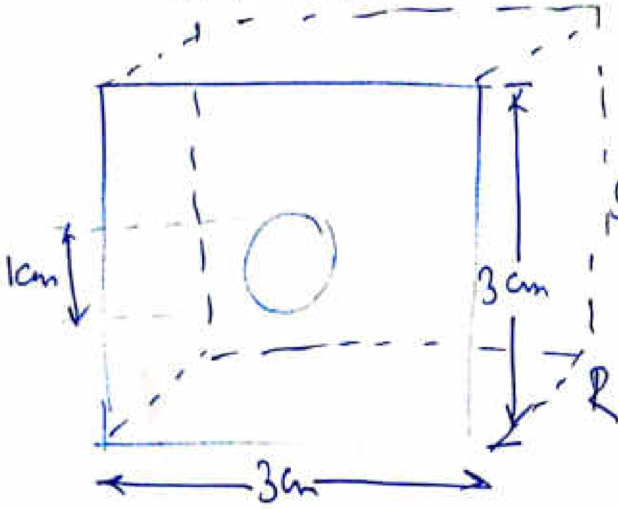
$$\text{or } \frac{1}{r^2}\frac{d}{dr}(r^2\, 5r^3) = \frac{\rho}{\epsilon_0}$$

$$\text{or } 25 r^2 = \frac{\rho}{\epsilon_0} \text{ or } \rho = 25\epsilon_0 r^2.$$

Charge contained within a sph ~~shell~~ surface of radius 'a'

$$Q = \int_0^a 4\pi r^2 dr\, \rho = 100\pi\epsilon_0 \int_0^a r^4 dr = 20\pi\epsilon_0 a^5.$$

Ex. A lead ($\sigma = 5\times10^{6}$ S/m). Find the resistance bet^n the sq ends.



$$R = \frac{l}{\sigma S}$$

$$S = d^2 - \pi r^2 = 3^2 - \pi\left(\frac{1}{2}\right)^2 = \left(9 - \frac{\pi}{4}\right) \text{ cm}^2$$

$$R = \frac{4}{5\times10^{6}\left(9-\pi/4\right)\times10^{-4}} = 974\,\Omega$$

Ex. Given the potential $V = \frac{10}{r^2}\sin\theta\cos\phi$

(a) Find the el flux den $\vec{D}$ at $(2, \pi/2, 0)$.

(b) Calculate the w.d. in moving a $10\mu C$ charge from point $A(1, 30^\circ, 120^\circ)$ to $B(4, 90^\circ, 60^\circ)$.

$$\vec{D} = \epsilon_0 \vec{E}$$

But $\vec{E} = -\vec{\nabla} V = -\left[\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi\right]$

$$= \frac{20}{r^3}\sin\theta\cos\phi\,\hat{a}_r - \frac{10}{r^3}\cos\theta\cos\phi\,\hat{a}_\theta + \frac{10}{r^3}\sin\phi\,\hat{a}_\phi$$

At $(2, \pi/2, 0)$,

$$\vec{D} = \epsilon_0 \vec{E}\left(r=2,\ \theta=\pi/2,\ \phi=0\right)$$

$$= \epsilon_0\left(\frac{20}{8}\hat{a}_r - 0\hat{a}_\theta + 0 a_\phi\right)$$

$$= 2.5\,\epsilon_0\,\hat{a}_r \ \text{C/m}^2 = 22.1\ a_r \ \text{pC/m}^2$$

(b) $W = -Q\int_A^B \vec{E}\cdot d\vec{l} = Q\,V_{AB} = Q(V_B - V_A)$

$$= 10\left(\frac{10}{16}\sin 90^\circ\cos 60^\circ - \frac{10}{1}\sin 30^\circ\cos 120^\circ\right)\cdot 10^{-6}$$

$$= 28.125 \ldots J$$

Ex. If $\vec{J} = \frac{1}{r^3}(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta)$ A/m² Calculate the current passing through

(a) A hemispherical shell of radius 20 cm, $0<\theta<\pi/2$, $0<\phi<2\pi$

(b) A spherical shell of radius 10 cm.

Solⁿ $I = \int \vec{J}\cdot d\vec{S}$ where $d\vec{S} = r^2 \sin\theta\, d\theta\, d\phi\, \hat{a}_r$

(a) $$I = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \frac{1}{r^3}\cdot 2\cos\theta\, r^2 \sin\theta\, d\phi\, d\theta \Big|_{r=0.2}$$

$$= \frac{2}{r}\cdot 2\pi \int_{\theta=0}^{\pi/2} \sin\theta\, d(\sin\theta)\Big|_{r=0.2}$$

$$= \frac{4\pi}{0.2}\,\frac{\sin^2\theta}{2}\Big|_0^{\pi/2} = 10\pi = 31.4 \text{ A}.$$

(b) $0 \le \theta \le \pi$ instead of $0 \le \theta \le \pi/2$ and $r = 0.1$ m.

$$I = \frac{4\pi}{0.1}\,\frac{\sin^2\theta}{2}\Big|_0^{\pi} = 0$$

$$I = \oint \vec{J}\cdot d\vec{S} = \int \vec{\nabla}\cdot\vec{J}\, dv = 0$$

Since $\vec{\nabla}\cdot\vec{J} = 0$

$$\vec{\nabla}\cdot\vec{J} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{2}{r}\cos\theta\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{1}{r^3}\sin^2\theta\right]$$

$$= -\frac{2}{r^4}\cos\theta + \frac{2}{r^4}\cos\theta = 0.$$