Tutorial

If $\vec{B}$ is uniform show that $\vec{A}(\vec{r}) = -\frac{1}{2}(\vec{r} \times \vec{B})$

The curl of $\vec{A} = -(\vec{r} \times \vec{B})/2$ is equal to

$$\vec{\nabla} \times \vec{A} = -\frac{1}{2}\vec{\nabla} \times (\vec{r} \times \vec{B}) = -\frac{1}{2}\left[(\vec{B}\cdot\vec{\nabla})\vec{r} - (\vec{r}\cdot\vec{\nabla})\vec{B} + \vec{r}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{r})\right]$$

Since $\vec{B}$ is uniform it is independent of r, θ, ϕ therefore 2nd & 3rd term on RHS are zero.

The 1st term in Cartesian coordinates is equal to

$$(\vec{B}\cdot\vec{\nabla})\vec{r} = \left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= B_x\hat{i} + B_y\hat{j} + B_z\hat{k} = \vec{B}$$

The 4th term, in Cartesian

$$\vec{B}(\vec{\nabla}\cdot\vec{r}) = \vec{B}\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z\right) = 3\vec{B}$$

Therefore, the curl of $\vec{A}$ is equal to

$$\vec{\nabla} \times \vec{A} = -\frac{1}{2}(\vec{B} - 3\vec{B}) = \vec{B}$$

The divergence of $\vec{A}$ is equal to

$$\vec{\nabla}\cdot\vec{A} = -\frac{1}{2}\vec{\nabla}\cdot(\vec{r} \times \vec{B}) = -\frac{1}{2}\left[\vec{B}\cdot(\vec{\nabla} \times \vec{r}) - \vec{r}\cdot(\vec{\nabla} \times \vec{B})\right] =$$

Three Fundamental Quantities of Magnetostatics

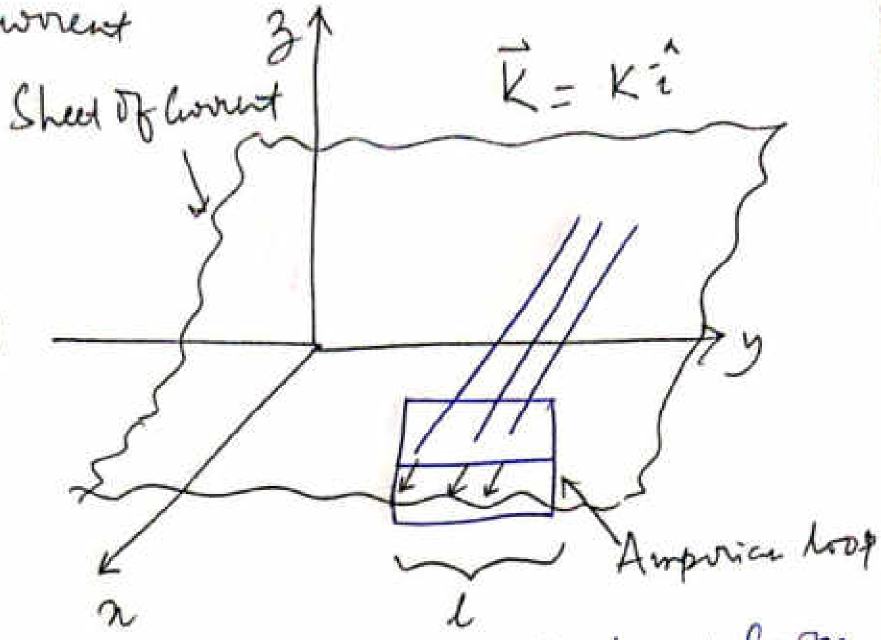
- The current density $\vec{J}$
- The mag fld $\vec{B}$
- The Vector potential $\vec{A}$

If One of them is known the other two can calculated.

Known ↓	$\vec{J}=$	$\vec{B}=$	$\vec{A}=$
$\vec{J}$		$\frac{\mu_0}{4\pi}\int \frac{\vec{J}\times\Delta\hat{r}}{\Delta r^2}d\tau$	$\frac{\mu_0}{4\pi}\int\frac{\vec{J}}{\Delta r}d\tau$
$\vec{B}$	$\frac{1}{\mu_0}\left(\vec{\nabla}\times\vec{B}\right)$		$\frac{1}{4\pi}\int\frac{\vec{B}\times\Delta\hat{r}}{\Delta r^2}d\tau$
$\vec{A}$	$-\frac{1}{\mu_0}\nabla^2\vec{A}$	$\vec{\nabla}\times\vec{A}$	

Scanned by CamScanner

2. Find the vector potential above and below the plane surface current



$$\vec{A} = -\frac{1}{2}(\vec{r} \times \vec{B})$$

Since the surface current extends to infinity we cannot use the surface integral of $\vec{K}/\Delta r$ to calculate $\vec{A}$ and an alternative method

In the region above the xy plane $(z>0)$ the mag fld is equal to

$$\vec{B} = -\frac{\mu_0}{2} K \hat{j}$$

Therefore,

$$\vec{A} = -\frac{1}{2}(\vec{r} \times \vec{B}) = -\frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 0 & -\frac{\mu_0}{2}K & 0 \end{vmatrix}$$

$$= -\frac{\mu_0}{4} K z \hat{i} + \frac{\mu_0}{4} K x \hat{k}$$

In the region below the xy plane $(z<0)$ the mag fld

$$\vec{B} = \frac{\mu_0}{2} K \hat{j}$$

Therefore,

$$\vec{A} = -\frac{1}{2}(\vec{r} \times \vec{B}) = -\frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 0 & \frac{\mu_0}{2}K & 0 \end{vmatrix} = \frac{\mu_0}{4} K z \hat{i} - \frac{\mu_0}{4} K x \hat{k}$$

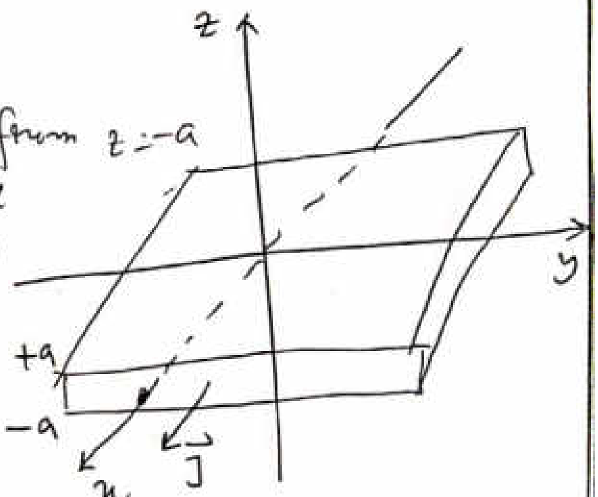
We can verify that by calculating $\vec{\nabla}\times\vec{A}$

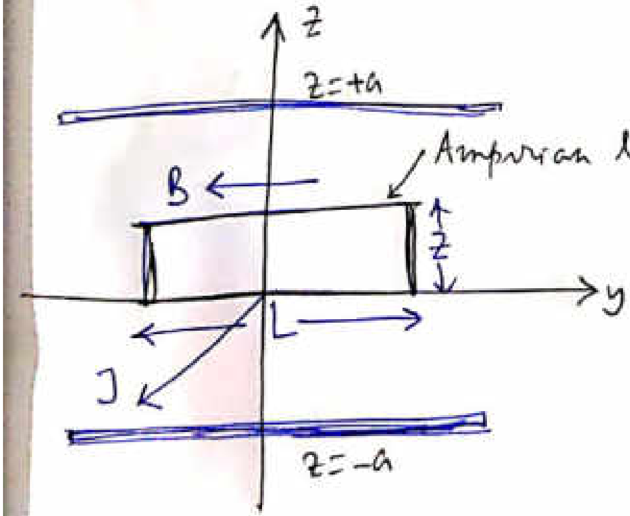
For $z>0$:

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{\mu_0}{4}Kz & 0 & \frac{\mu_0}{4}Kz \end{vmatrix} = -\frac{\mu_0}{2}K\hat{j} = \vec{B}$$

The vector potential $\vec{A}$ is however not uniquely defined. for ex: $\vec{A} = -\frac{\mu_0}{2}Kz\,\hat{i}$ and $\vec{A} = \frac{\mu_0}{2}Kx\,\hat{k}$ are also possible sol^n that generate the same mag fld. These sol^n also satisfy the requirement that $\vec{\nabla}\cdot\vec{A} = 0$.

3. A thick slab extending from $z=-a$ to $z=+a$ carries a uniform vol current $\vec{J} = J\hat{i}$. Find the mag fld as a f^n of z, both inside and outside the slab.





Symmetry

mag fld $z>0$

$\Downarrow$ mirror image

mag fld $z<0$

In the Amperian loop the mag fld $=0$ in $z=0$

Current is flowing out of the paper

dir^n of $d\vec{a}$ to be $\parallel^{el}$ to the dir^n of $\vec{J}$

Therefore $\int_{\text{Surface}} \vec{J}\cdot d\vec{a} = JzL \qquad 0<z<a$

$$\int_{\text{Surface}} \vec{J}\cdot d\vec{a} = JaL \qquad z>a$$

The dirn of evaluation of the line integral of $\vec{B}$ must be consistent with our choice of the dirn of $d\vec{a}$ (right-hand rule). This requires that the line integral of $\vec{B}$ must be evaluated in a counter-clockwise dir? The line integral of $\vec{B}$ is equal to

$$\oint_{\text{Line}} \vec{B}\cdot d\vec{l} = BL$$

Applying Ampere's law we obtain $\vec{B}$:

$$B = \frac{\mu_0}{L}\int_{\text{Surface}} \vec{J}\cdot d\vec{a} = \mu_0 Jz \qquad 0<z<a$$

$$B = \frac{\mu_0}{L}\int_{\text{Surface}} \vec{J}\cdot d\vec{a} = \mu_0 Ja \qquad z>a$$

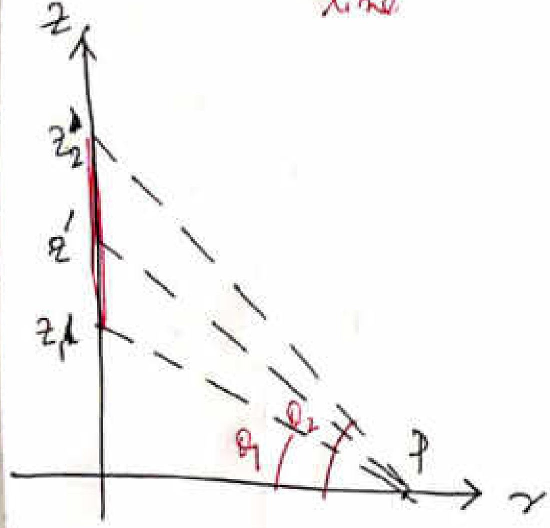
Thus $B(z) = -\mu_0 Ja\hat{j} \qquad a<z$

$$B(z) = -\mu_0 Jz\hat{j} \qquad -a<z<a$$

$$B(z) = \mu_0 Ja\hat{j} \qquad z<-a$$

4. Find the magnetic vector potential of a finite segment of straight wire, carrying a current I.

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{line} \frac{\vec{I}}{\Delta r}\,dl = \frac{\mu_0 I}{4\pi}\int_{line} \frac{d\vec{l}}{\Delta r} \quad \text{for line current}$$



The current at infinity is zero, therefore we can use the expression for $\vec{A}$ in terms of the line integral of the current I. Consider the wire located along the z axis betn z_1 and z_2. and use cylindrical coord. The vector potential at a pt-P is independent of ϕ (cylindrical symmetry) and equal to

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{line} \frac{d\vec{l}}{\Delta r} = \frac{\mu_0 I}{4\pi}\int_{z_1}^{z_2} \frac{dz'}{\sqrt{r^2+z'^2}}\hat{k} = \frac{\mu_0 I}{4\pi}\ln\left[\frac{z_2+\sqrt{r^2+z_2^2}}{z_1+\sqrt{r^2+z_1^2}}\right]\hat{k}$$

Here we have assumed that the origin of the coordinate system is chosen such that P has $z=0$

$$\vec{B} = \vec{\nabla}\times\vec{A} = -\frac{\partial A_z}{\partial r}\hat{\phi} = -\frac{\mu_0 I}{4\pi}\left[\frac{r}{\sqrt{r^2+z_2^2}}\,\frac{1}{z_2+\sqrt{r^2+z_2^2}} - \frac{r}{\sqrt{r^2+z_1^2}}\,\frac{1}{z_1+\sqrt{r^2+z_1^2}}\right]\hat{\phi}$$

$$= \frac{\mu_0 I}{4\pi r}\left[\frac{z_2}{\sqrt{r^2+z_2^2}} - \frac{z_1}{\sqrt{r^2+z_1^2}}\right]\hat{\phi} = \frac{\mu_0 I}{4\pi r}\left[\sin\theta_2 - \sin\theta_1\right]\hat{\phi}$$