Matrices

Lecture-1

In a basis $\hat{e}_i$ both vectors and linear operators can be described in terms of their components with respect to the basis. These components may be displayed as an array of numbers called a <u>matrix</u>.

If a linear operator A transforms vectors from an N-dim vector space, for which we choose a basis $\hat{e}_j$, $j=1,\dots,N$, into vectors belonging to an M-dim vector space, with basis f_i, $i=1,\dots M$, then we may represent the operator A by the matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & & & \\ A_{M1} & A_{M2} & \cdots & A_{MN} \end{pmatrix}$$

Component A_{ij} is also denoted by $(A)_{ij}$

$$\sum_j (A+B)_{ij} x_j = \sum_j A_{ij} x_j + \sum_j B_{ij} x_j$$

$$\sum_j (\lambda A)_{ij} x_j = \lambda \sum_j A_{ij} x_j$$

$$\sum_j (AB)_{ij} x_j = \sum_k A_{ik} (Bx)_k = \sum_j \sum_k A_{ik} B_{kj} x_j .$$

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

$$(\lambda A)_{ij} = \lambda A_{ij}$$

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}$$

Matrix addition and multiplication by a scalar

$$S_{ij} = A_{ij} + B_{ij}$$

$$A + B = B + A$$

matrix addition is commutative and associative.

Multiplication by a scalar is distributive and associative.

$$\vec{y} = A\vec{x}$$

$$y_i = \sum_{j=1}^{N} A_{ij} x_j \quad \text{for } i = 1, \ldots, M$$

$$y = Ax$$

$$\begin{pmatrix} y_1 \\ \boxed{y_2} \\ \vdots \\ y_M \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ \hline A_{21} & A_{22} & \cdots & A_{2N} \\ \hline \vdots & \vdots & \ddots & \vdots \\ A_{M1} & A_{M2} & \cdots & A_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

$$y_2 = A_{21} x_1 + A_{22} x_2 + \cdots + A_{2N} x_N$$

If instead we operate with A on a basis e_j having all components zero except for the jth, which equals unity

$$A e_j = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & & \ddots & \vdots \\ A_{M1} & A_{M2} & \cdots & A_{MN} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} A_{1j} \\ \boxed{A_{2j}} \\ \vdots \\ A_{Mj} \end{pmatrix}$$

$$P_{ij} = \sum_{k=1}^{N} A_{ik} B_{kj} \quad \text{for } i = 1, \ldots, M; \; j = 1, \ldots, R$$

$$AB \neq BA$$

multiplication of matrix is not in general commutative.

$$(A+B)C = AC + BC$$

$$C(A+B) = CA + CB$$

The null and identity matrices

$$AO = O = OA$$

$$A+O = O+A = A$$

The identity matrix I has the property

$$AI = IA = A$$

identity matrix must be square

$$I_N = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

Transpose of a matrix

$$(AB)^T = B^T A^T$$

$$(ABC\cdots G)^T = G^T \cdots C^T B^T A^T.$$

$$(AB)^T_{ij} = (AB)_{ji} = \sum_k A_{jk} B_{ki}$$

$$= \sum_k (A^T)_{kj} (B^T)_{ik}$$

$$= \sum_k (B^T)_{ik} (A^T)_{kj}$$

$$= (B^T A^T)_{ij}$$

The Complex and Hermitian conjugates of a matrix

Complex Conjugate A^*

Hermitian " $A^\dagger$

The Complex conjugate of a matrix A is the matrix obtained by taking the Complex Conjugate of each of the elements of A, i.e.

$$\left(A^*_{ij}\right) = \left(A_{ij}\right)^*$$

if a matrix is real (i.e. it Contains only real elements) then $A^* = A$

The Hermitian Conjugate (or adjoint) of a matrix A is simply the transpose of its complex conjugate, or equv. the complex Conjugate of its transpose, i.e.

$$A^\dagger = \left(A^*\right)^T = \left(A^T\right)^*$$

$$(AB \cdots G)^\dagger = G^\dagger \cdots B^\dagger A^\dagger$$

if A is real (so that $A^* = A$) then $A^\dagger = A^T$, and so the Hermitian conjugate may be considered as a generalization of the transpose to complex matrices.

inner product of two vectors. Suppose that in a given orthonormal basis the vectors $\vec{a}$ and $\vec{b}$ may be represented by the Column matrices

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix}$$

aking the Hermitian Conjugate of a, to give a row matrix, and multiplying by b we obtain

$$a^{\dagger} b = \begin{pmatrix} a_1^* & a_2^* & \cdots & a_N^* \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix} = \sum_{i=1}^{N} a_i^* b_i$$

inner product $\langle a | b \rangle$

for real vectors $a^{T} b = \sum_{i=1}^{N} a_i b_i$

If the basis $\hat{e}_i$ is not orthonormal, so that in gen:

$$\langle \hat{e}_i | \hat{e}_j \rangle = G_{ij} \neq \delta_{ij}$$

$$\langle \vec{a} | \vec{b} \rangle = \langle a_1 \hat{e}_1 + \cdots + a_N \hat{e}_N | b_1 \hat{e}_1 + \cdots + b_N \hat{e}_N \rangle$$

$$= \sum_{i=1}^{N} a_i^* b_i \langle \hat{e}_i | \hat{e}_i \rangle + \sum_{i} \sum_{j \neq i} a_i^* b_j \langle \hat{e}_i | \hat{e}_j \rangle$$

$$= \sum_{i=1}^{N} a_i^* b_i$$

Then if $\vec{a} = \sum_{i=1}^{N} a_i \hat{e}_i$ and $\vec{b} = \sum_{i=1}^{N} b_i \hat{e}_i$

$$\langle \vec{a} | \vec{b} \rangle = \left\langle \sum_{i=1}^{N} a_i \hat{e}_i \middle| \sum_{j=1}^{N} b_j \hat{e}_j \right\rangle$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} a_i^* b_j \langle \hat{e}_i | \hat{e}_j \rangle$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} a_i^* G_{ij} b_j .$$

in gen where the base vectors $\hat{e}_1, \hat{e}_2, \ldots \hat{e}_N$ are not orthonormal (or orthogonal)

$$G_{ij} = \langle \hat{e}_i | \hat{e}_j \rangle$$

$G_{ij} = G^*_{ji}$ $\quad ||\vec{a}|| = \langle \vec{a} | \vec{a} \rangle$ is real, since the

$$\langle \vec{a} | \vec{a} \rangle^* = \sum_{i=1}^{N} \sum_{j=1}^{N} a_i \, G^*_{ij} \, a^*_j = \sum_{j=1}^{N} \sum_{i=1}^{N} a^*_j \, G_{ji} \, a_i = \langle \vec{a} | \vec{a} \rangle$$

the scalar product of $\vec{a}$ and $\vec{b}$ in terms of their components with respect to this basis

$$\langle \vec{a} | \vec{b} \rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} a^*_i \, G_{ij} \, b_j = a^\dagger G \, b$$

where G is the $N \times N$ matrix with ~~the~~ elements G_{ij}.