Lecture-I

Electric fields in Material Space

Conduction Currents

How the electric field behaves in a conductor or dielectric, it is appropriate to consider electric current. Electric current is generally caused by the motion of electric charges.

$$I = \frac{dQ}{dt}$$

current density $\vec{J}$, $J = \frac{\Delta I}{\Delta S}$ or $\Delta I = J \Delta S$

If current density is not normal to the surface

$$\Delta I = \vec{J} \cdot \Delta \vec{S}$$

Thus, the total current flowing through a surface S is

$$I = \int_S \vec{J} \cdot d\vec{S}$$

Conduction current requires a conductor. A conductor is characterized by a large no of free elⁿ that provide condⁿ current due to an impressed el. fld. When an el fld $\vec{E}$ is applied, the force on an e^- with charge $-e$ is

$$\vec{F} = -e\vec{E}$$

Since the elⁿ is not in free space, it will not experience an avg accⁿ under the influence of the el fld. Rather it suffers constant collisions with the atomic lattice and drifts from one atom to another.

$$\frac{m\vec{u}}{\tau} = -e\vec{E} \quad \text{or} \quad \vec{u} = -\frac{e\tau}{m}\vec{E}$$

$\tau \rightarrow$ avg time interval betⁿ collisions.

If there are n el^n per unit vol, the el^n charge density

$$\rho_v = -ne$$

Thus the conduction current density

$$\vec{J} = \rho_v \vec{u} = \frac{ne^2\tau}{m}\vec{E} = \sigma\vec{E}$$

or $\boxed{J = \sigma E}$, $\sigma = \frac{ne^2\tau}{m}$ conductivity of the conductor

point form of Ohm's law.

A perfect conductor ($\sigma = \infty$) cannot contain an electrostatic field within it.

A conductor is called an equipotential body, implying that the potential is the same everywhere in the conductor.

$$\vec{E} = -\vec{\nabla}V = 0$$

Continuity Eq^n and Relaxation Time

From the principle of charge conservation, the time rate of decrease of charge within a given volume must be equal to the net outward current flow through the surface of the volume

$$I_{out} = \oint \vec{J}\cdot d\vec{s} = \frac{-dQ_{in}}{dt}$$

Div th $\oint_S \vec{J}\cdot d\vec{s} = \int_V \vec{\nabla}\cdot\vec{J}\, dv$

But $-\frac{dQ_{in}}{dt} = -\frac{d}{dt}\int \rho_v\, dv = -\int_V \frac{\partial\rho_v}{\partial t}\, dv$

$$\int_V \vec{\nabla}\cdot\vec{J}\, dv = -\int_V \frac{\partial\rho_v}{\partial t}\, dv$$

or $\boxed{\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho_v}{\partial t}}$ Continuity eq^n

there can be no accumulation of charge at any point.

For steady currents, $\frac{\partial \rho_v}{\partial t} = 0$, $\vec{\nabla} \cdot \vec{J} = 0$, showing that the total charge leaving a volume is the same as the total charge entering it.

$$\vec{J} = \sigma \vec{E}$$

Gauss's law $\quad \vec{\nabla} \cdot \vec{E} = \frac{\rho_v}{\varepsilon}$

$$\vec{\nabla} \cdot \sigma \vec{E} = \frac{\sigma \rho_v}{\varepsilon} = -\frac{\partial \rho_v}{\partial t}$$

$$\text{or} \quad \frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon} \rho_v = 0$$

$$\text{or} \quad \frac{\partial \rho_v}{\rho_v} = -\frac{\sigma}{\varepsilon} \partial t$$

Integrating $\quad \ln \rho_v = -\frac{\sigma t}{\varepsilon} + \ln \rho_{v_0}$

$$\rho_v = \rho_{v_0} e^{-t/\tau_r} \quad \text{where } T_r = \frac{\varepsilon}{\sigma}$$

time constant

The time constant T_r is known as the relaxation time or rearrangement time. Relaxation time is the time it takes a charge placed in the interior of a material to drop to $e^{-1} = (36.8\%)$ of its initial value.

It is short for good conductors and long for good dielectrics.

$$Cu: \quad \sigma = 5.8 \times 10^{7} \text{ S/m} \quad \varepsilon_r = 1$$

$$T_r = \frac{\varepsilon_r \varepsilon_0}{\sigma} = 1 \times \frac{10^{-9}}{36\pi} \times \frac{1}{5.8 \times 10^{7}}$$

$$= 1.53 \times 10^{-19} \text{ s.}$$

showing a rapid decay of charge placed inside copper. This implies that for good conductors, the relaxation time is so short that most of the charge will vanish from any interior pt. and appear the surface (as surface charge) almost instantaneously.

fused quartz $\sigma = 10^{-17}$ S/m, $\varepsilon_r = 5.0$

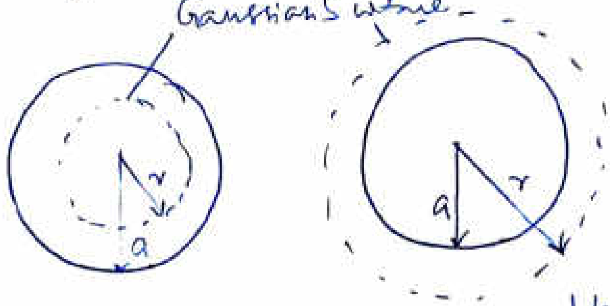
$$T_r = 5 \times \frac{10^{-9}}{36\pi} \times \frac{1}{10^{-17}}$$

$$= 51.2 \text{ days}$$

showing a very large relaxation time.

Uniformly Charged Sphere

To determine $\vec{D}$



Uniform charge ρ_0 C/m^3

$r \leq a$ $\qquad$ $r \geq a$

$$Q_{enc} = \int_v \rho_v \, dv = \rho_0 \int_v dv = \rho_0 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \rho_0 \frac{4}{3} \pi r^3$$

$$\Psi = \oint_S \vec{D} \cdot d\vec{S} = D_r \oint_S dS = D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$= D_r \cdot 4\pi r^2$$

Hence $\Psi = Q_{enc}$ gives

$$D_r \cdot 4\pi r^2 = \frac{4\pi r^3}{3} \rho_0$$

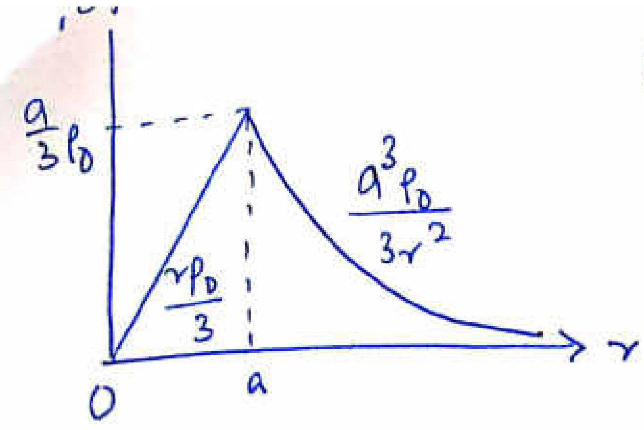
or $\vec{D} = \frac{r}{3} \rho_0 \vec{a}_r \qquad 0 < r \leq a$

For $r \geq a$

$$Q_{enc} = \int_v \rho_v \, dv = \rho_0 \int_v dv = \rho_0 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \rho_0 \cdot \frac{4}{3} \pi a^3$$

while $\Psi = \oint \vec{D} \cdot d\vec{S} = D_r \cdot 4\pi r^2$



Hence $D_r \, 4\pi r^2 = \frac{4}{3}\pi a^3 \rho_0$

or $\vec{D} = \frac{a^3}{3r^2}\rho_0 \, \vec{a_r} \qquad r \geq a$

$$\vec{D} = \begin{cases} \frac{r}{3}\rho_0 \vec{a_r}, & 0 < r \leq a \\ \frac{a^3}{3r^2}\rho_0 \vec{a_r} & r \geq a \end{cases}$$

* $\vec{D}$ must be constant on the Gaussian surface.

Ex1. Given that $\vec{D} = 2\rho \cos^2\phi \, \vec{a_z}$ C/m^2. Calculate the charge density at $(1, \pi/4, 3)$ and the total charge enclosed by the cylinder of radius 1 m with $2 \leq z \leq 2$

$$\rho_v = \vec{\nabla} \cdot \vec{D} = \frac{\partial D_z}{\partial z} = \rho \cos^2\phi$$

At $(1, \pi/4, 3)$, $\rho_v = 1 . \cos^2(\pi/4) = 0.5 \ C/m^3$

$$Q = \int_v \rho_v \, dv = \int_v \rho \cos^2\phi \; \rho \, d\phi \, d\rho \, dz$$

$$= \int_{z=-2}^{2} dz \int_{\phi=0}^{2\pi} \cos^2\phi \, d\phi \int_{\rho=0}^{1} \rho^2 \, d\rho$$

$$= 4(\pi)(1/3)$$

$$= \frac{4\pi}{3} \ C$$

Ex. 2. If $\vec{D} = (2y^2+z)\vec{a}_x + 4xy\vec{a}_y + x\vec{a}_z$ C/m^2 find

(a) the vol. charge density at $(-1, 0, 3)$

(b) the flux through the cube defined by $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$

(c) the total charge enclosed by the cube.

Ex. 3. A charge dist^n with sph sym has density

$$\rho_v = \begin{cases} \dfrac{\rho_0 r}{R} & 0 \le r \le R \\ 0 & r > R \end{cases}$$

Determine $\vec{E}$ everywhere.

$$\varepsilon_0 \oint_S \vec{E} \cdot d\vec{s} = Q_{enc} = \int_V \rho_v \, dv$$

(a) For $r < R$

$$\varepsilon_0 E_r 4\pi r^2 = Q_{enc} = \int_{r=0}^{r}\int_{\theta=0}^{\pi}\int_{0}^{2\pi} \rho_v r^2 \sin\theta \, d\phi \, d\theta \, dr$$

$$= \int_0^r 4\pi r^2 \frac{\rho_0 r}{R} dr = \frac{\rho_0 \pi r^4}{R}$$

$$\therefore \vec{E} = \frac{\rho_0 r^2}{4\varepsilon_0 R}\vec{a}_r$$

(b) For $r > R$

$$\varepsilon_0 E_r 4\pi r^2 = Q_{enc} = \int_0^r\int_0^{\pi}\int_0^{2\pi} \rho_v r^2 \sin\theta \, d\phi \, d\theta \, dr$$

$$= \int_0^R \frac{\rho_0 r}{R} 4\pi r^2 dr + \int_R^r 0 \cdot 4\pi r^2 dr$$

$$= \pi \rho_0 R^3$$

$$\therefore \vec{E} = \frac{\rho_0 R^3}{4\pi r^2}\vec{a}_r$$