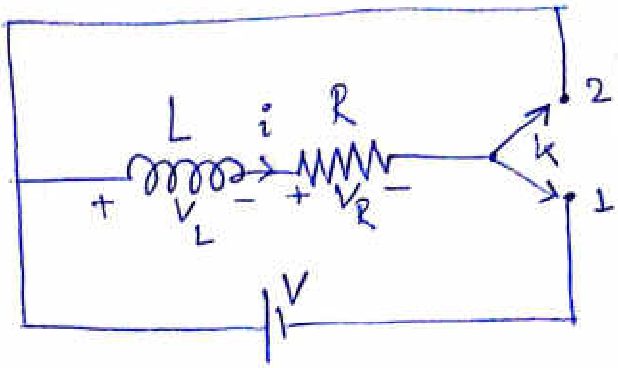
Transient Phenomena in Electric Circuits

Growth and Decay of Currents in series LR Circuits

(i) Growth of current

induction of back emf $-L\frac{di}{dt}$

opposes the driving emf V



$$i = \frac{V - L\frac{di}{dt}}{R} \quad \Rightarrow L\frac{di}{dt} + Ri = V$$

$$\Rightarrow \frac{di}{i - V/R} = -\frac{R}{L}dt$$

· Assuming that $i = 0$ at $t = 0$

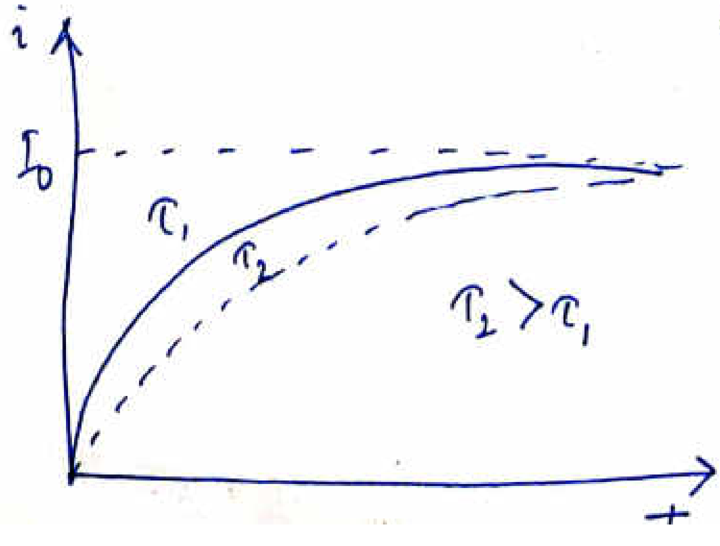
$$\int_0^i \frac{di}{i - V/R} = -\frac{R}{L}\int_0^t dt$$

$$\Rightarrow \ln\frac{i - V/R}{-V/R} = -\frac{R}{L}t$$

$$\Rightarrow i = \frac{V}{R}\left(1 - e^{-R/L\, t}\right)$$

$$= I_0\left(1 - e^{-R/L\, t}\right)$$

$$\frac{di}{dt} = I_0\frac{R}{L}e^{-R/L\, t} = \frac{R}{L}(I_0 - i)$$



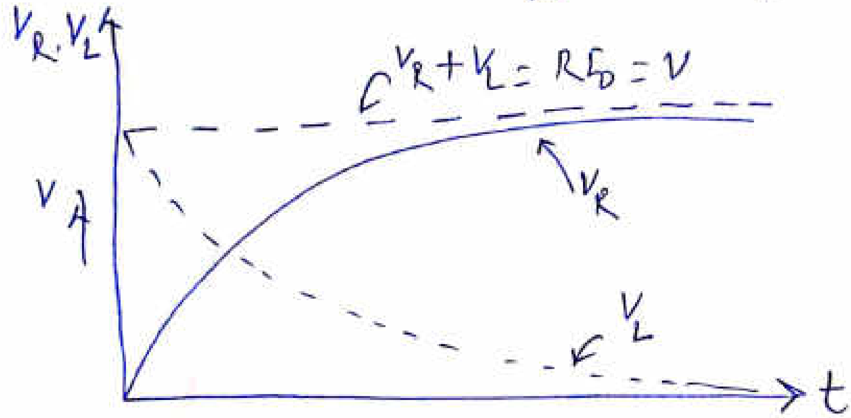
At $t = \tau$,

$$i = I_0\left(1 - \frac{1}{e}\right) \approx 0.63 I_0$$

When the current reaches its max^m steady value the variation of current with time stops and so the back emf becomes zero. At this stage the current behaves as if there were no inductor in the circuit. The instantaneous voltage drops across L and R are given by

$$V_R = Ri = RI_0\left(1 - e^{-R/L\,t}\right)$$

$$\text{and } V_L = L\frac{di}{dt} = RI_0\, e^{-R/L\,t}$$



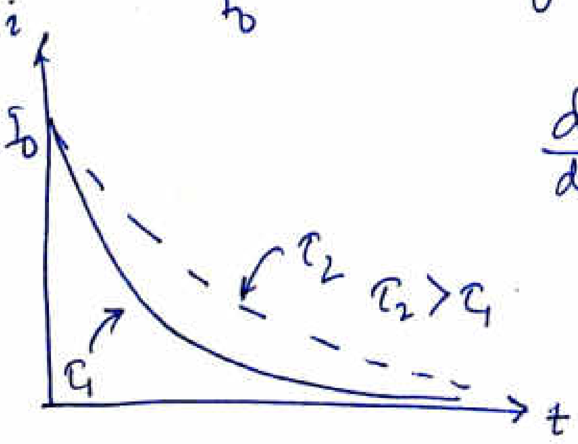
(i) Decay of current

$$i = \frac{0 - L\frac{di}{dt}}{R} \Rightarrow L\frac{di}{dt} + Ri = 0 \Rightarrow \frac{di}{i} = -\frac{R}{L}dt$$

Assuming that $i = I_0$ at $t = 0$

$$\int_{I_0}^{i} \frac{di}{i} = -\frac{R}{L}\int_0^t dt \Rightarrow \ln\frac{i}{I_0} = -\frac{R}{L}t$$

$$\Rightarrow i = I_0 e^{-R/L\,t}$$



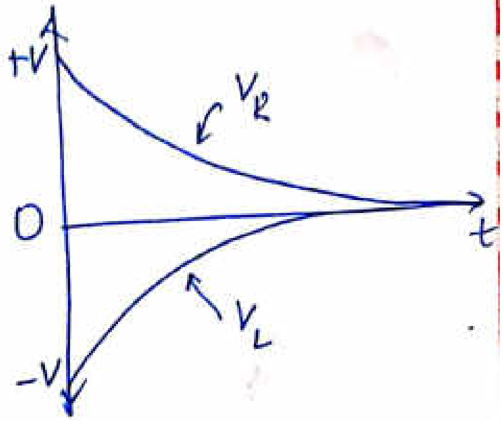
$$\frac{di}{dt} = -\frac{R}{L}I_0 e^{-R/L\,t} = -\frac{I_0}{\tau}e^{-t/\tau}$$

$$\Rightarrow \left.\frac{di}{dt}\right|_{t=0} = -\frac{I_0}{\tau}$$

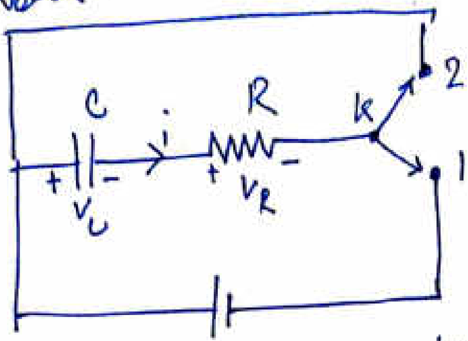
at $t = \tau$, $i = I_0/e \approx 0.37 I_0$.

$$v_L = L\frac{di}{dt} = -RI_0 e^{-R/L\, t} = -Ve^{-R/L\, t}$$

$$v_R = Ri = RI_0 e^{-R/L\, t} = Ve^{-R/L\, t}$$



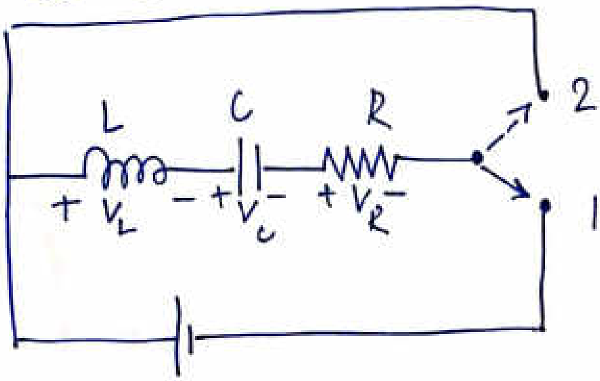
Homework:



$$q = q_0\left(1-e^{-t/CR}\right)$$

$$q = q_0 e^{-t/CR}$$

Series LRC Circuit



$$L\frac{di}{dt} + \frac{q}{C} + Ri = V$$

$$i = \frac{dq}{dt}$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V$$

or $$\frac{d^2q}{dt^2} + 2b\frac{dq}{dt} + \omega_0^2\left(q - \frac{V}{L\omega_0^2}\right) = 0$$

$2b = \frac{R}{L}$

$\omega_0^2 = \frac{1}{LC}$

$Q = q - V/L\omega_0^2$

$$\Rightarrow \frac{d^2Q}{dt^2} + 2b\frac{dQ}{dt} + \omega_0^2 Q = 0$$

Trial sol^n: $Q = Ae^{\alpha t}$

$$(\alpha^2 + 2b\alpha + \omega_0^2)Ae^{\alpha t} = 0$$

$$\alpha^2 + 2b\alpha + \omega_0^2 = 0$$

$$\alpha_1 = -b + \sqrt{b^2 - \omega_0^2} \quad \text{and} \quad \alpha_2 = -b - \sqrt{b^2 - \omega_0^2}$$

$$Q = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}$$

$$= e^{-bt}\left[A_1 e^{\sqrt{b^2-\omega_0^2}\,t} + A_2 e^{-\sqrt{b^2-\omega_0^2}\,t}\right]$$

$$\therefore\ q = Q + \frac{V}{L\omega_0^2} = q_0 + e^{-bt}\left[A_1 e^{pt} + A_2 e^{-pt}\right]$$

$q_0 = V/L\omega_0^2 = V.C$ and $p = \sqrt{b^2 - \omega_0^2}$

$t = 0,\ q = 0,\ i = \frac{dq}{dt} = 0$

$0 = q_0 + A_1 + A_2$ and $0 = (A_1 - A_2)p - (A_1 + A_2)b$

$$A_1 = -\frac{q_0}{2}\left(1 + \frac{b}{p}\right)$$

$$A_2 = -\frac{q_0}{2}\left(1 - \frac{b}{p}\right)$$

Complete soln: $q = q_0 - \frac{q_0}{2}\cdot e^{-bt}\left[\left(1+\frac{b}{p}\right)e^{pt} + \left(1-\frac{b}{p}\right)e^{-pt}\right]$

Case I: $b^2 > \omega_0^2$ $\quad p = \sqrt{b^2 - \omega_0^2}$

Case II: $b^2 = \omega_0^2$

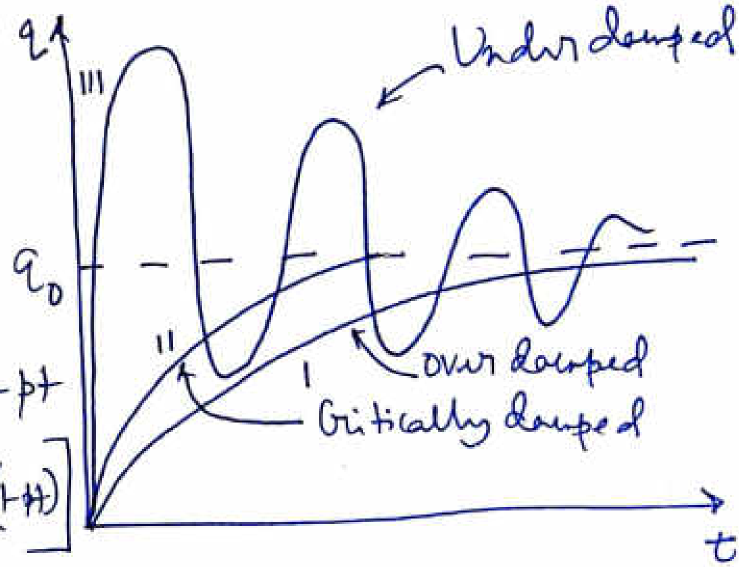
$p \to 0$, b/p tends to ∞

we assume $p \ll 1$

$e^{pt} \simeq 1 + pt$ and $e^{-pt} = 1 - pt$

$q = q_0 - \frac{q_0}{2}e^{-bt}\left[\left(1+\frac{b}{p}\right)(1+pt) + \left(1-\frac{b}{p}\right)(1-pt)\right]$

$= q_0 - q_0(1+bt)e^{-bt}$ $\qquad b^2 = \omega_0^2 \therefore R = 2\sqrt{L/C}$ Critical damping resistance



Case III: $b^2 < \omega_0^2$

$p = \sqrt{b^2 - \omega_0^2}$ is imaginary

$p = j\sqrt{\omega_0^2 - b^2} = j\omega$, $\omega = \sqrt{\omega_0^2 - b^2}$

$$q = q_0 - \frac{q_0}{2}e^{-bt}\left[\left(e^{j\omega t} + e^{-j\omega t}\right) + \frac{b}{j\omega}\left(e^{j\omega t} - e^{-j\omega t}\right)\right]$$

$$= q_0 - q_0 e^{-bt}\left[\cos\omega t + \frac{b}{\omega}\sin\omega t\right]$$

$$= q_0 - \frac{q_0}{\omega}e^{-bt}\left[\omega\cos\omega t + b\sin\omega t\right]$$

putting $\omega = A\cos\theta$, $b = A\sin\theta$

$A = \sqrt{\omega^2 + b^2} = \omega_0$ and $\theta = \tan^{-1}\frac{b}{\omega}$ $\qquad q = q_0 - q_0\frac{\omega_0}{\omega}e^{-bt}\cos(\omega t - \theta)$

freq^{y} of Osc^{n}

$$f = \frac{\omega}{2\pi} = \frac{\sqrt{\omega_0^2 - b^2}}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$i = \frac{dq}{dt} = \frac{q_0}{\omega} b e^{-bt}\left[\omega \cos\omega t + b \sin\omega t\right] - \frac{q_0}{\omega} e^{-bt}\left[-\omega^2 \sin\omega t + b\omega \cos\omega t\right]$$

$$= \frac{q_0 \omega_0^2}{\omega} e^{-bt} \sin\omega t$$

R becomes negligible then $b = 0$, $\omega = \omega_0$

$$q = q_0 - q_0 \cos\omega_0 t \qquad \omega_0 \rightarrow \text{undamped natural freq}^{y}$$

$$i = q\omega_0 \sin\omega_0 t$$

Homework: Discharging of the Capacitor

Prob: Examine whether the discharge of a charged capacitor $0.1\mu f$ through an inductive coil of inductance 100mH and resistance 200Ω is oscillatory or not. If it is osc^{ry}, find the freq^{y} of Osc^{n}.

$$R = 2\sqrt{\frac{L}{C}} = 2\sqrt{\frac{100\times10^{-3}}{0.1\times10^{-6}}} = 2000\Omega$$

As the coil resistance is smaller than this value the discharge will be osc^{ry}. The freq^{y} of Osc^{n}

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \frac{1}{2\pi}\sqrt{\frac{1}{0.1\times0.1\times10^{-6}} - \frac{200^2}{4\times(0.1)^2}}$$

$$= 1584\ Hz.$$

 DB

Topic : Transient Phenomena in Electric Circuits

(ii) Discharging of the capacitor

Source is cut off and the stored charge on the capacitor begins to discharge through L & R.

$$L\frac{di}{dt}+\frac{q}{c}+Ri=0$$

$$\text{or}\quad L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{c}q=0 \qquad 2b=R/L$$

$$\omega_0^2=1/LC$$

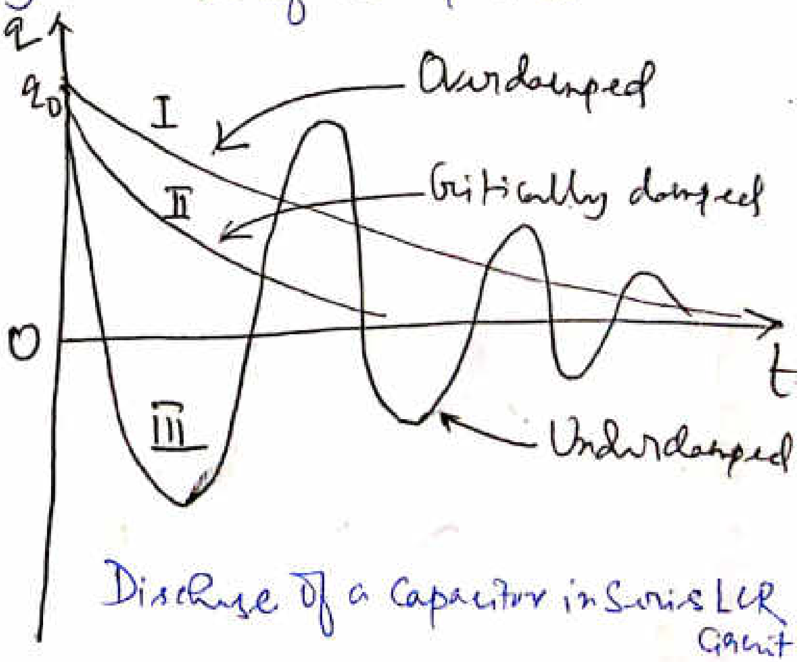
$$\text{or}\quad \frac{d^2q}{dt^2}+2b\frac{dq}{dt}+\omega_0^2 q=0$$

$$q=e^{-bt}\left[A_1e^{pt}+A_2e^{-pt}\right] \qquad p=\sqrt{b^2-\omega_0^2}$$

$$q=\frac{q_0}{2}e^{-bt}\left[\left(1+\frac{b}{p}\right)e^{pt}+\left(1-\frac{b}{p}\right)e^{-pt}\right] \qquad A_1=\frac{q_0}{2}\left(1+\frac{b}{p}\right)$$

$$A_2=\frac{q_0}{2}\left(1-\frac{b}{p}\right)$$

Case I : p : +ve non oscillatory discharge of the capacitor



Discharge of a Capacitor in Series LCR Circuit

Case II : $b^2=\omega_0^2$

$p=0$, b/p tends to ∞

$p<<1$, $e^{\pm pt}\approx 1\pm pt$

$$q=q_0(1+bt)e^{-bt}$$

Case III : $b^2<\omega_0^2$

$p=\sqrt{b^2-\omega_0^2}$ is imaginary

$p=j\sqrt{\omega_0^2-b^2}=j\omega,\ \omega=\sqrt{\omega_0^2-b^2}$

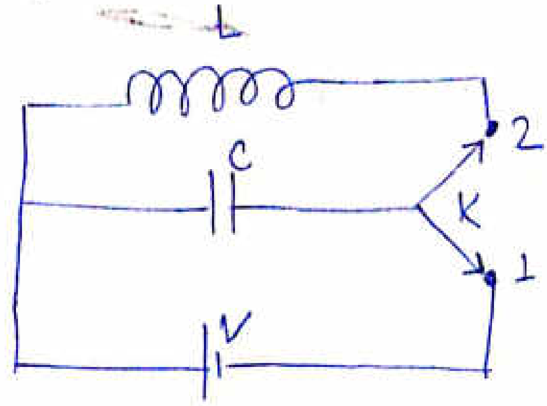
$$q=\frac{q_0\omega_0}{\omega}e^{-bt}\cos(\omega t-\theta),\ \theta=\tan^{-1}\frac{b}{\omega}.$$

Oscillatory!

$$f=\frac{\omega}{2\pi}=\frac{\sqrt{\omega_0^2-b^2}}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}$$

$$i=\frac{dq}{dt}=-\frac{q_0\omega_0^2}{\omega}e^{-bt}\sin\omega t$$

Discharge of a Capacitor Through a Pure Inductor and Electrical Oscillations



$$\omega_0^2 = \frac{1}{LC}$$

$$L\frac{di}{dt} + \frac{q}{C} = 0 \Rightarrow L\frac{d^2q}{dt^2} + \frac{q}{C} = 0 \Rightarrow \frac{d^2q}{dt^2} + \omega_0^2 = 0$$

$$q = A_1 \cos\omega_0 t + A_2 \sin\omega_0 t$$

at $t=0$, $q = q_0 = VC$ and $\frac{dq}{dt} = 0$; $q_0 = A_1$ & $A_2 = 0$

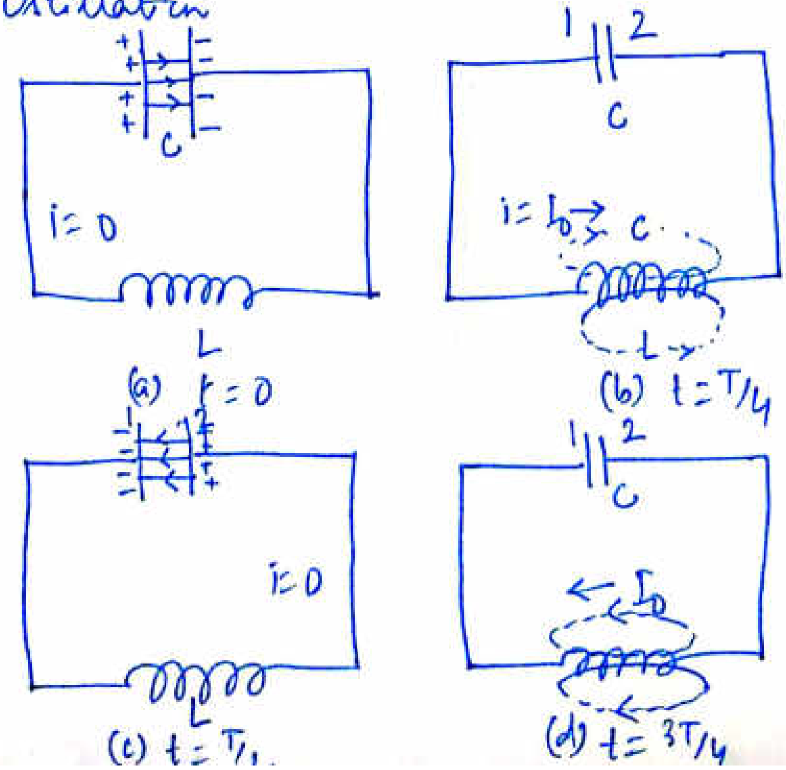
$$q = q_0 \cos\omega_0 t$$

Thus the charge on the Capacitor varies in an Oscillatory manner with const amplitude q_0 and $f = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$

$$i = \frac{dq}{dt} = -q_0\omega_0 \sin\omega_0 t$$

$$\Rightarrow i = q_0\omega_0 \cos(\omega_0 t + \pi/2)$$

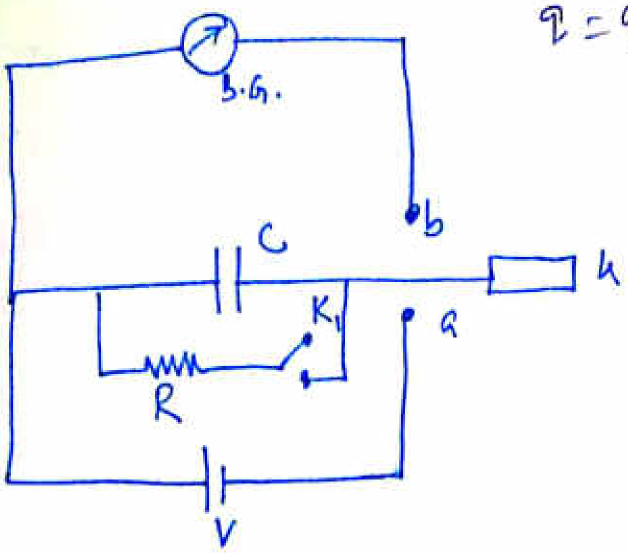
Electric Oscillation



electrostatic en $q_0^2/2C$

magnetic en $\frac{1}{2}Li^2$

Measurement of High Resistance by leakage



$$q = q_0 e^{-t/CR} \quad \therefore R = \frac{t}{C \ln \frac{q_0}{q}}$$

first ballistic throw d_0 is noted

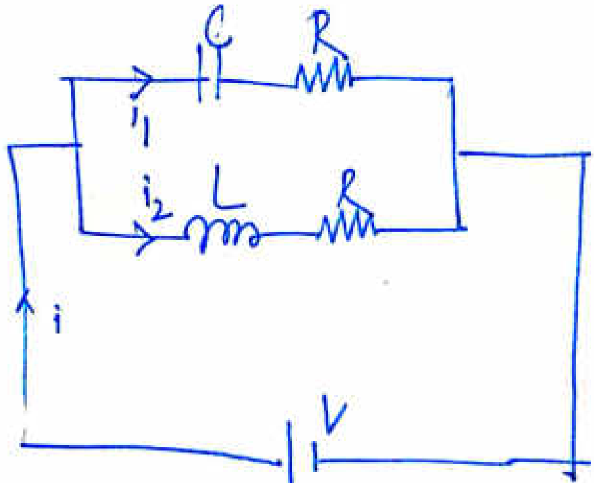
$$q_0/q = d_0/d$$

$$R = \frac{t}{C \ln d_0/d}$$

$$R_0 = \frac{t_1}{C \ln \frac{d_0}{d}}$$

$$\frac{1}{R} = \frac{1}{R_a} + \frac{1}{R_0}$$ natural leakage resistance

Prob:



If $L/R = CR$ Show that the current drawn from the source is indep of time.

at any time t is

$$i = i_1 + i_2$$

$$q = q_0\left(1 - e^{-t/CR}\right) \text{ where } q_0 = CV$$

$$\therefore i_1 = \frac{dq}{dt} = \frac{q_0}{CR} e^{-t/CR} = \frac{V}{R} e^{-t/CR}$$

also $i_2 = \frac{V}{R}\left(1 - e^{-R/L\, t}\right) = \frac{V}{R}\left(1 - e^{-t/CR}\right)$

Therefore $i = i_1 + i_2 = \frac{V}{R}$

which is indep of t.