Sem-II Conductors   Lecture Notes- III

Continuity Eq^n & Relaxation time

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho_v}{\partial t}$$

For steady currents $\frac{\partial \rho_v}{\partial t} = 0$ & hence $\vec{\nabla}\cdot\vec{J} = 0$
total charge leaving a vol is the same as the total charge entering it. Kirchhoff's current law follows from this.

Ohm's law $\vec{J} = \sigma\vec{E}$

Gauss's law $\vec{\nabla}\cdot\vec{E} = \frac{\rho_v}{\varepsilon}$

$$\vec{\nabla}\cdot\sigma\vec{E} = \frac{\sigma\rho_v}{\varepsilon} = -\frac{\partial \rho_v}{\partial t}$$

or $$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon}\rho_v = 0$$

Homogeneous linear ordinary diff^{nl} eq^n

$$\frac{\partial \rho_v}{\rho_v} = -\frac{\sigma}{\varepsilon}\partial t$$

integrating both sides

$$\ln \rho_v = -\frac{\sigma t}{\varepsilon} + \ln \rho_{v_0}$$

$$\rho_v = \rho_{v_0} e^{-t/\tau} \quad \text{where } \tau = \frac{\varepsilon}{\sigma}$$

time constant

ρ_{v_0} is the initial charge density (ρ_v at $t=0$)

introduction of charge at some interior pt. of the material results in a decay of vol. charge density ρ_v. Associated with the decay is charge movement from the interior pt. at which it was introduced to the surface of the material.

Errr $\tau \rightarrow$ relaxation time or rearrangement time.

short for good conductors

& long " good dielectrics

$Cu \rightarrow \quad \sigma = 5.8 \times 10^{7}$ S/m , $\varepsilon_r = 1$

$$\tau = \frac{\varepsilon_r \varepsilon_0}{\sigma} = 1 \times \frac{10^{-9}}{36\pi} \times \frac{1}{5.8 \times 10^{7}}$$

$$= 1.53 \times 10^{-19} \text{ s.}$$

rapid decay of charge placed inside copper.

vanish from any interior pt. and appear at the surface almost instantaneously.

for fused quartz, $\sigma = 10^{-17}$ S/m , $\varepsilon_r = 5.0$

$$\tau = 5 \times \frac{10^{-9}}{36\pi} \times \frac{1}{10^{-17}} = 51.2 \text{ days.}$$

One may consider the introduced charge to remain wherever placed for times up to days.

Electrostatic Boundary Value Problems.

Poisson's & Laplace's eq^n are easily derived from Gauss's law (for a linear isotropic material medium)

$$\vec{\nabla} \cdot \vec{D} = \nabla \cdot \varepsilon \vec{E} = \rho_v \quad \& \quad \vec{E} = -\vec{\nabla} V$$

$$\vec{\nabla} \cdot (-\varepsilon \vec{\nabla} V) = \rho_v$$ for an inhomogeneous medium

for homogeneous medium $\boxed{\nabla^2 V = -\frac{\rho_v}{\varepsilon}}$ Poisson's eq^n.

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$\rho_v = 0$ (charge free region)

$$\boxed{\nabla^2 V = 0} \quad \text{Laplace's eq}^n$$

Cylindrical $\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$

Spherical $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$

Uniqueness Theorem   analytical, graphical, numerical, expt. etc.

There is only one sol^n [Unique]

Any sol^n of Laplace's eq^n that satisfies the same boundary cond^n must be the only sol^n regardless of the method used. This is known as the uniqueness theorem. The theorem applies to any sol^n of Poisson's or Laplace's eq^n in a given region or closed surface.

Can be proved by contradiction

We assume that there are two sol^n V_1 and V_2 of Laplace's eq^n, both of which satisfy the prescribed b.c. Thus

$$\nabla^2 V_1 = 0, \qquad \nabla^2 V_2 = 0$$

$$V_1 = V_2 \quad \text{on the boundary}$$

We consider their difference

$$V_d = V_2 - V_1$$

which obeys $\nabla^2 V_d = \nabla^2 V_2 - \nabla^2 V_1 = 0$

$V_d = 0$ on the boundary

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From the div. the.

$$* \int_V \vec{\nabla}.\vec{A}\, dv = \oint_S \vec{A}.d\vec{S}$$

We let $\vec{A} = V_d \vec{\nabla} V_d$ and use a vector identity

$$\vec{\nabla}.\vec{A} = \vec{\nabla}.\left(V_d \vec{\nabla} V_d\right) = V_d \nabla^2 V_d + \vec{\nabla} V_d . \vec{\nabla} V_d$$

But $\vec{\nabla}^2 V_d = 0$; so $\vec{\nabla}.\vec{A} = \vec{\nabla} V_d . \vec{\nabla} V_d$

$$* \int_V \vec{\nabla} V_d . \vec{\nabla} V_d \, dv = \oint_S V_d \vec{\nabla} V_d . d\vec{S}$$

r.h.s. vanishes Hence: $\int_V |\vec{\nabla} V_d|^2 dv = 0$

Since the int^n is always +ve

$$\vec{\nabla} V_d = 0$$

or $V_d = V_2 - V_1 =$ Const everywhere in V

$V_1 = V_2$ at the boundary Hence $V_d = 0$ or $V_1 = V_2$ everywhere, showing that V_1 and V_2 cannot be diff^{nt} sol^n of the same prob.

Resistance & Capacitance

A perfect conductor ($\sigma = \infty$) cannot contain an electrostatic field within it.

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$$\vec{F} = -e\vec{E}$$

Since the e^- is not in free space, it will not experience an avg accn under the influence of the el fld.

$$\frac{m\vec{u}}{\tau} = -e\vec{E}$$

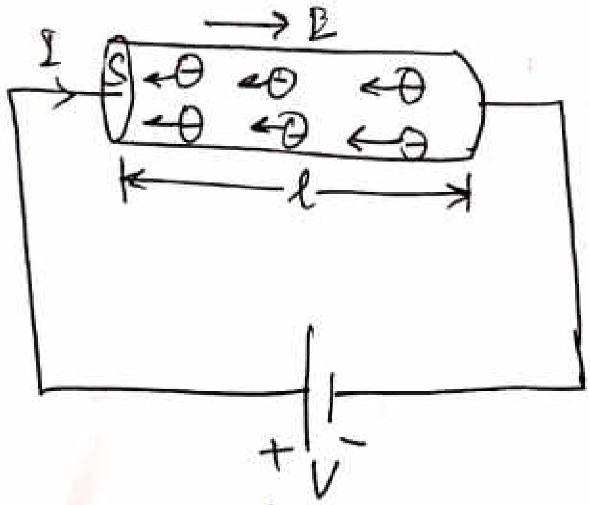
$$\Rightarrow \vec{u} = -\frac{e\tau}{m}\vec{E}$$

$$\rho_v = -ne$$

Cond current density $\vec{J} = \rho_v \vec{u} = \frac{ne^2\tau}{m}\vec{E} = \sigma\vec{E}$

$$\Rightarrow \boxed{J = \sigma E} \qquad \sigma = \frac{ne^2\tau}{m}$$

point form of Ohm's law



$\vec{E} = 0$, $\rho_v = 0$, $V_{ab} = 0$ inside a condr

$$E = \frac{V}{l}$$

$$J = \frac{I}{S}$$

$$R = \frac{V}{I} = \frac{\int \vec{E} \cdot d\vec{l}}{\int_S \sigma \vec{E} \cdot d\vec{s}}$$

$$\frac{I}{S} = \sigma E = \frac{\sigma V}{l}$$

$$R = \frac{V}{I} = \frac{l}{\sigma S}$$

$$P = \int_V \vec{E} \cdot \vec{J} \, dv$$

$$\Rightarrow R = \frac{\rho_c l}{S}, \quad \rho_c = \frac{1}{\sigma}$$

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